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Simple K3 singularities which are hypersurface sections of toric singularities.

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Let N be a free $\mathbf{Z}$-module of rank $n+1$. Let $\mathcal{E}^n$ be the set of pairs (σ, u_0) consisting of an $(n+1)$-dimensional cone in $N_{\mathbb{R}}$ and a point u_0 in σ satisfying the following conditions (G) and (E), respectively.

(G) There exists the point $v(\sigma)$ in N^* such that σ is generated by finite elements in $\{ u \in N \mid \langle v(\sigma), u\rangle = 1 \}$.

(E) $\dim \Delta_\sigma(u_0) = n$ and $v(\sigma) \in \mathrm{Int}(\Delta_\sigma(u_0))$, where $\Delta_\sigma(u_0)$ is the convex hull of $\{ v \in \sigma^* \cap N^* \mid \langle v, u_0\rangle = 1 \}$.

Note that if an $(n+1)$-dimensional cone σ in $N_{\mathbb{R}}$ satisfies the condition (G), then the point $v(\sigma)$ is unique and σ is strongly convex rational cone. Let (σ, u_0) be a pair in $\mathcal{E}^n$ and let $f = \sum_{v\in\Delta_\sigma(u_0)\cap N^*} c_v z^v$ + higher term $\in \mathbb{C}[\sigma^* \cap N^*]$, for certain non-zero coefficients c_v. In the previous paper[2], we show that if f is non-degenerate and the hypersurface section $X = \{f = 0\}$ of $Y = \mathrm{Spec}\,\mathbb{C}[\sigma^* \cap N^*]$ defined by f has an isolated singularity at $y := \mathrm{orb}(\sigma)$, then (X, y) is a purely elliptic of $(0,n-1)$-type singularity (for the definition, see [3]). Especially, when $n = 3$, (X,y) is a simple K3 singularity. We also show in [2] that $\mathcal{E}^3$ is a finite set, where $\mathcal{E}^n$ is the set of equivalence classes of pairs in $\tilde{\mathcal{E}}^n$ by the following equivalence relation. $(\sigma,u_0) \sim (\sigma',u_0')$ if and only if there exists an element g in $GL(N)$ such that $g_{\mathbb{R}}\sigma = \sigma'$ and that $g_{\mathbb{R}}(u_0) = u_0'$. In this paper, we show that $\mathcal{E}^n$ is a finite set for each integer n greater than 2.

Let $\tilde{\mathcal{F}}^n = \{ (\sigma,u_0) \in \tilde{\mathcal{E}}^n \mid u_0 \in N \}$ and let $\mathcal{F}^n$ be the set of the equivalence classes of the pairs in $\tilde{\mathcal{F}}^n$.

Proposition 1. There exists a map π from $\mathcal{E}^n$ to $\mathcal{F}^n$ such that $\pi^{-1}(\alpha)$ is a finite set for each α in $\mathcal{F}^n$.

Proof. Let (σ,u_0) be in $\tilde{\mathcal{E}}^n$. Then u_0 is in $N_{\mathbb{Q}}$, by the condition (E). Hence the module $N(u_0)$ generated by N and u_0 is also a free $\mathbb{Z}$-module of rank $n+1$. Therefore, there exists an isomorphism g from $N(u_0)$ to N. First, we show that the pair $(g_{\mathbb{R}}\sigma, g(u_0))$ is in $\tilde{\mathcal{F}}^n$.

Since $\langle v(\sigma), u_0\rangle = 1$, $v(\sigma)$ is in ${}^tg(N^*) = N(u_0)^* = \{ v \in N^* \mid \langle v,u_0\rangle \in \mathbb{Z} \}$. Hence ${}^tg_{\mathbb{R}}^{-1}(v(\sigma))$ is in N^*. Therefore, $g_{\mathbb{R}}\sigma$ satisfies (G) and $v(g_{\mathbb{R}}\sigma) = {}^tg_{\mathbb{R}}^{-1}(v(\sigma))$. Since $\{ v \in \sigma^* \cap N^* \mid \langle v,u_0\rangle = 1 \} = \{ v \in \sigma^* \cap N(u_0)^* \mid \langle v,u_0\rangle = 1 \} = {}^tg(\{ v' \in (g_{\mathbb{R}}\sigma)^* \cap N^* \mid \langle v',g(u_0)\rangle = 1 \}$, we see that

$\Delta_{g_{\mathbb{R}}\sigma}(g(u_0)) = {}^tg_{\mathbb{R}}^{-1}(\Delta_\sigma(u_0))$. Hence $g(u_0)$ satisfies (E).

we easily see that if $(\sigma,u_0) \sim (\sigma',u_0')$, then $(g_{\mathbb{R}}\sigma,g(u_0)) \sim (g_{\mathbb{R}}'\sigma',g'(u_0'))$, for any isomorphisms $g : N(u_0) \simeq N$ and $g' : N(u_0') \simeq N$. We denote by π, the map from $\mathcal{E}^n$ to $\mathcal{F}^n$ thus obtained. Next, we show that $\pi^{-1}(\alpha)$ is a finite set for each eqivalence class α in $\mathcal{F}^n$.

Let β and β' be elements in $\pi^{-1}(\alpha)$, let (σ,u_0), (σ',u_0') and (τ,t_0) be representatives of β, β' and α, respectively. Then there exist isomorphisms $g : N(u_0) \simeq N$ and $g' : N(u_0') \simeq N$ such that $g_{\mathbb{R}}\sigma = g_{\mathbb{R}}'\sigma' = \tau$ and that $g(u_0) = g'(u_0') = t_0$. Assume that $g(N) = g'(N)$. Then the map $h := (g')^{-1}_{|g(N)}\cdot g_{|N}$ is in $GL(N)$, $h_{\mathbb{R}}\sigma = \sigma'$ and $h_{\mathbb{R}}(u_0) = u_0'$. Hence $(\sigma,u_0) \sim (\sigma',u_0')$. On the other hand, $\langle v(\sigma),g_{\mathbb{R}}^{-1}(u)\rangle = \langle {}^tg_{\mathbb{R}}^{-1}(v(\sigma)),u\rangle = \langle v(\tau),u\rangle = 1$ for primitive elements u in all 1-dimensional faces of τ. Since $g_{\mathbb{R}}^{-1}(u)$ are generators of 1-dimensional faces of σ, $g_{\mathbb{R}}^{-1}(u) \in N$, by the condition (G). Hence $g(N)$ contains the module N' generated by primitive elements in all 1-dimensional faces of τ. Since τ is an $(n+1)$-dimensional rational cone, N' is also a free $\mathbb{Z}$-module of rank $n+1$. Hence N/N' is a finite group. Therefore, $\#(\pi^{-1}(\alpha)) \leq \#\{\text{ subgroups of } N/N' \} < +\infty$. q.e.d.

Note that $\{ u \in \mathrm{Int}(\sigma) \cap N \mid \langle v(\sigma),u\rangle = 1 \} = \{u_0\}$ for any pair (σ,u_0) in $\tilde{\mathcal{F}}^n$ (see [2, Proposition 1.8]). Hence we have an injective map from $\mathcal{F}^n$ to $\mathcal{P}^n := \tilde{\mathcal{P}}^n/\sim$, where $\tilde{\mathcal{P}}^n$

is the set of n-dimensional integral convex polytopes P in $\mathbb{R}^n$ with $\mathrm{Int}(P) \cap \mathbb{Z}^n = \{0\}$ and $P \sim P'$ if and only if there exists an element g in GL(n,$\mathbb{Z}$) such that $g_{\mathbb{R}}P = P'$. Hence if $\mathcal{P}^n$ is finite, then $\mathcal{E}^n$ is also finite, by the above proposition.

Proposition 2. $\mathcal{P}^n$ is a finite set.

Proof. There exists a real number L such that vol(P) < L for any P in $\tilde{\mathcal{P}}^n$, by [1]. Let S be the set of simplices $\overline{0v_1v_2 \dots v_n}$ spanned by 0, $v_1 = {}^t(p_{11},0,\dots,0)$, ... , $v_j = {}^t(p_{j1},\dots,p_{jj},0,\dots,0)$ ... and $v_n = {}^t(p_{n1},\dots,p_{nn})$ in $\mathbb{Z}^n$ such that $0 \leqq p_{jk} < p_{jj}$ for j = 1 through n and for k = 1 through j-1 and that $p_{11}p_{22} \cdots p_{nn} < n!L$. Clearly, S is a finite set. Let P be in $\tilde{\mathcal{P}}^n$. Then P contains n vertices u_1, u_2, ... and u_n which are linearly independent. There exists an element g in GL(n,$\mathbb{Z}$) such that $g(u_j) = (p_{j1}, \dots, p_{jj}, 0, \dots, 0)$ ($0 \leqq p_{jk} < p_{jj}$ for k = 1 through j-1). Since $\mathrm{vol}(\overline{0u_1 \dots u_n}) \leqq \mathrm{vol}(P) < L$, $g(\overline{0u_1 \dots u_n}) \in S$. On the other hand, each point u in P is a linear combination $a_1u_1 + a_2u_2 + \dots + a_nu_n$ of u_1, u_2, ... and u_n. If $a_j \neq 0$, then $L > \mathrm{vol}(P) \geqq \mathrm{vol}(\overline{0u_1 \dots u_{j-1}uu_{j+1} \dots u_n}) = |a_j|\mathrm{vol}(\overline{0u_1 \dots u_n}) = |a_j|p_{11}p_{22} \cdots p_{nn}/n!$. Hence g(P) is contained in the compact set $C := \{ a_1g(u_1) + a_2g(u_2) + \dots + a_ng(u_n) \mid |a_j| \leqq$

$n!L/p_{11}\cdots p_{nn}$ }. Since the set of integral convex polytopes contained in C is finite, $\mathcal{F}^n$ is also finite. q.e.d.

Thus we obtain:

Theorem 3. $\mathcal{E}^n$ is finite.

References

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[3] K. Watanabe, Minimal resolution of a quasi-homogeneous simple K3 singularity, in this volume.